Weighted Hermite-Hadamard-Type Inequalities by Identities Related to Generalizations of Steffensen's Inequality

Pečarić, Josip; Perušić Pribanić, Anamarija; Smoljak Kalamir, Ksenija

Source / Izvornik: Mathematics, 2022, 10

Journal article, Published version Rad u časopisu, Objavljena verzija rada (izdavačev PDF)

https://doi.org/10.3390/math10091505

Permanent link / Trajna poveznica: https://urn.nsk.hr/urn:nbn:hr:157:855657

Rights / Prava: Attribution 4.0 International/Imenovanje 4.0 međunarodna

Download date / Datum preuzimanja: 2025-03-23

mage not found or type unknown Repository / Repozitorij:



Repository of the University of Rijeka, Faculty of Civil Engineering - FCERI Repository









Article Weighted Hermite–Hadamard-Type Inequalities by Identities Related to Generalizations of Steffensen's Inequality

Josip Pečarić¹, Anamarija Perušić Pribanić^{2,*} and Ksenija Smoljak Kalamir³

- ¹ Croatian Academy of Sciences and Arts, Trg Nikole Šubića Zrinskog 11, 10000 Zagreb, Croatia; pecaric@element.hr
- ² Faculty of Civil Engineering, University of Rijeka, Radmile Matejčić 3, 51000 Rijeka, Croatia
- ³ Faculty of Textile Technology, University of Zagreb, Prilaz Baruna Filipovića 28a, 10000 Zagreb, Croatia; ksmoljak@ttf.hr
- * Correspondence: anamarija.perusic@uniri.hr

Abstract: In this paper, we obtain some new weighted Hermite–Hadamard-type inequalities for (n + 2)–convex functions by utilizing generalizations of Steffensen's inequality via Taylor's formula.

Keywords: weighted Hermite–Hadamard inequality; Steffensen's inequality; Taylor's formula; *n*-convex functions

MSC: 26D15; 26A51

1. Introduction

The Hermite–Hadamard inequality is one of the most important mathematical inequalities. It was discovered independently first by Hermite [1] and later by Hadamard [2]. The classical Hermite–Hadamard inequality provides an estimate from below and above the mean value of convex function $f: [a, b] \rightarrow \mathbb{R}$. More precisely, we have the following.

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a} \int_a^b f(x) \, dx \le \frac{f(a)+f(b)}{2}$$

To illustrate the importance of the Hermite–Hadamard inequality, let us mention that the Hermite–Hadamard inequality can be considered as the necessary and sufficient condition for convexity of a function. Furthermore, the Hermite–Hadamard inequality has an important role in numerical analysis, mathematical analysis and functional analysis. Various generalizations, extensions and applications of the Hermite-Hadamard inequality have appeared in the literature (see [3–8]).

In this paper, we consider the weighted Hermite–Hadamard inequality for convex functions given in following theorem (see [8–10]).

Theorem 1. Let $p: [a, b] \to \mathbb{R}$ be a non-negative function. If $f: [a, b] \to \mathbb{R}$ is a convex function, then we have the following:

$$f(m) \le \frac{1}{P(b)} \int_a^b p(x) f(x) dx \le \frac{b-m}{b-a} f(a) + \frac{m-a}{b-a} f(b)$$

$$P(b)f(m) \le \int_{a}^{b} p(x)f(x)dx \le P(b)\left[\frac{b-m}{b-a}f(a) + \frac{m-a}{b-a}f(b)\right],\tag{1}$$

where the following is the case.

$$P(t) = \int_a^t p(x) dx$$
 and $m = \frac{1}{P(b)} \int_a^b p(x) x dx$.



Citation: Pečarić, J.; Perušić Pribanić, A.; Smoljak Kalamir, K. Weighted Hermite-Hadamard-Type Inequalities by Identities Related to Generalizations of Steffensen's Inequality. *Mathematics* **2022**, *10*, 1505. https://doi.org/10.3390/math10091505

Academic Editor: Jaan Janno

Received: 30 March 2022 Accepted: 29 April 2022 Published: 1 May 2022

Publisher's Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.



Copyright: © 2022 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https:// creativecommons.org/licenses/by/ 4.0/).

0ľ

In 1918, Steffensen proved the following inequality (see [11]).

Theorem 2 ([11]). Suppose that f is non-increasing and g is integrable on [a, b] with $0 \le g \le 1$ and $\lambda = \int_a^b g(t) dt$. Then, we have the following.

$$\int_{b-\lambda}^{b} f(t)dt \le \int_{a}^{b} f(t)g(t)dt \le \int_{a}^{a+\lambda} f(t)dt.$$
(2)

The inequalities are reversed for f non-decreasing.

Many papers have been devoted to generalizations and refinements of Steffensen's inequality and its connection to other well-known inequalities such as Gauss–Steffensen's, Hölder's, Jenssen–Steffensen's and other inequalities. A complete overview of the results related to Steffensen's inequality can be found in monographs [12,13].

By using the Mitrinović [14] result in which the inequalities in (2) follow from identities:

$$\int_{a}^{a+\lambda} f(t)dt - \int_{a}^{b} f(t)g(t)dt$$
$$= \int_{a}^{a+\lambda} [f(t) - f(a+\lambda)][1 - g(t)]dt + \int_{a+\lambda}^{b} [f(a+\lambda) - f(t)]g(t)dt$$

and

$$\int_{a}^{b} f(t)g(t)dt - \int_{b-\lambda}^{b} f(t)dt$$
$$= \int_{a}^{b-\lambda} [f(t) - f(b-\lambda)]g(t)dt + \int_{b-\lambda}^{b} [f(b-\lambda) - f(t)][1-g(t)]dt$$

and using Taylor's formulae in points *a* and *b*

$$f(x) = \sum_{i=0}^{n-1} \frac{f^{(i)}(a)}{i!} (x-a)^i + \frac{1}{(n-1)!} \int_a^x f^{(n)}(t) (x-t)^{n-1} dt$$
$$f(x) = \sum_{i=0}^{n-1} \frac{f^{(i)}(b)}{i!} (x-b)^i - \frac{1}{(n-1)!} \int_x^b f^{(n)}(t) (x-t)^{n-1} dt$$

in paper [15], the authors proved the following identities related to generalizations of Steffensen's inequality.

Theorem 3 ([15]). Let $f: [a, b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \ge 2$ and let $g: [a, b] \to \mathbb{R}$ be an integrable function such that $0 \le g \le 1$. Let $\lambda = \int_a^b g(t) dt$ and let the function G_1 be defined by the following.

$$G_1(x) = \begin{cases} \int_a^x (1 - g(t)) dt, & x \in [a, a + \lambda], \\ \int_x^b g(t) dt, & x \in [a + \lambda, b]. \end{cases}$$

Then, we have the following:

$$\int_{a}^{a+\lambda} f(t)dt - \int_{a}^{b} f(t)g(t)dt + \sum_{i=0}^{n-2} \frac{f^{(i+1)}(a)}{i!} \int_{a}^{b} G_{1}(x)(x-a)^{i}dx$$

$$= -\frac{1}{(n-2)!} \int_{a}^{b} \left(\int_{t}^{b} G_{1}(x)(x-t)^{n-2}dx\right) f^{(n)}(t)dt$$
(3)

and the following is obtained.

$$\int_{a}^{a+\lambda} f(t)dt - \int_{a}^{b} f(t)g(t)dt + \sum_{i=0}^{n-2} \frac{f^{(i+1)}(b)}{i!} \int_{a}^{b} G_{1}(x)(x-b)^{i}dx$$

$$= \frac{1}{(n-2)!} \int_{a}^{b} \left(\int_{a}^{t} G_{1}(x)(x-t)^{n-2}dx \right) f^{(n)}(t)dt.$$
(4)

Theorem 4 ([15]). Let $f: [a, b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \ge 2$ and let $g: [a, b] \to \mathbb{R}$ be an integrable function such that $0 \le g \le 1$. Let $\lambda = \int_a^b g(t) dt$ and let the function G_2 be defined by the following.

$$G_2(x) = \begin{cases} \int_a^x g(t)dt, & x \in [a, b - \lambda], \\ \int_x^b (1 - g(t))dt, & x \in [b - \lambda, b]. \end{cases}$$

Then, we have the following:

$$\int_{a}^{b} f(t)g(t)dt - \int_{b-\lambda}^{b} f(t)dt + \sum_{i=0}^{n-2} \frac{f^{(i+1)}(a)}{i!} \int_{a}^{b} G_{2}(x)(x-a)^{i}dx$$

$$= -\frac{1}{(n-2)!} \int_{a}^{b} \left(\int_{t}^{b} G_{2}(x)(x-t)^{n-2}dx\right) f^{(n)}(t)dt$$
(5)

and the following is obtained.

$$\int_{a}^{b} f(t)g(t)dt - \int_{b-\lambda}^{b} f(t)dt + \sum_{i=0}^{n-2} \frac{f^{(i+1)}(b)}{i!} \int_{a}^{b} G_{2}(x)(x-b)^{i}dx$$

$$= \frac{1}{(n-2)!} \int_{a}^{b} \left(\int_{a}^{t} G_{2}(x)(x-t)^{n-2}dx \right) f^{(n)}(t)dt.$$
(6)

Since, in this paper, we will deal with n-convex functions, let us recall the definition of the n-convex function. For more details on convex functions, we refer the interested reader to [6,8].

Let *f* be a real-valued function defined on the segment [a, b]. The *divided difference* of order *n* of the function *f* at distinct points $x_0, ..., x_n \in [a, b]$ is defined recursively (see [8]) by the following. $f[x_i] = f(x_i), (i = 0, ..., n)$

$$f[x_0, \dots, x_n] = \frac{f[x_1, \dots, x_n] - f[x_0, \dots, x_{n-1}]}{x_n - x_0}.$$

The value $f[x_0, ..., x_n]$ is independent of the order of the points $x_0, ..., x_n$. The definition may be extended to include the case in which some (or all) of the points coincide. Assuming that $f^{(j-1)}(x)$ exists, we define the following.

$$f[\underbrace{x,\ldots,x}_{j-times}] = \frac{f^{(j-1)}(x)}{(j-1)!}.$$

Definition 1 ([8]). A function $f : [a, b] \to \mathbb{R}$ is said to be *n*-convex on [a, b], $n \ge 0$, if for all choices of (n + 1) distinct points in [a, b], the n - th order divided difference of f satisfies the following.

$$f[x_0, ..., x_n] \ge 0$$

Note that 1–convex functions are non-decreasing functions and 2–convex functions are convex functions. An n–convex function need not to be n–times differentiable; how-

ever, if $f^{(n)}$ exists, then f is n-convex if and only if $f^{(n)} \ge 0$. The following property also holds: if f is an (n + 2)-convex function, then there exists the n-th derivative $f^{(n)}$, which is a convex function.

The aim of this paper is to use identities related to generalizations of Steffensen's inequality, obtained by using Taylor's formula, to prove new weighted Hermite–Hadamard-type inequalities for (n + 2)–convex functions.

2. Main Results

In this section, applying identities given in Theorems 3 and 4 and the properties of n-convex functions, we derive new weighted Hermite–Hadamard-type inequalities.

Theorem 5. Let $f: [a, b] \to \mathbb{R}$ be (n + 2)-convex on [a, b] and $f^{(n-1)}$ absolutely continuous for $n \ge 2$. Let $g: [a, b] \to \mathbb{R}$ be an integrable function such that $0 \le g \le 1$ and $\lambda = \int_a^b g(t) dt$. Let function G_1 be defined by the following.

$$G_{1}(x) = \begin{cases} \int_{a}^{x} (1 - g(t)) dt, & x \in [a, a + \lambda], \\ \int_{x}^{b} g(t) dt, & x \in [a + \lambda, b]. \end{cases}$$
(7)

Then, we have the following:

$$P_{1}(b) \cdot f^{(n)}(m_{1}) \leq (n-2)! \left[\int_{a}^{b} f(t)g(t)dt - \int_{a}^{a+\lambda} f(t)dt - \sum_{i=0}^{n-2} \frac{f^{(i+1)}(a)}{i!} \int_{a}^{b} G_{1}(x)(x-a)^{i}dx \right]$$
(8)
$$\leq P_{1}(b) \cdot \left[\frac{b-m_{1}}{b-a} f^{(n)}(a) + \frac{m_{1}-a}{b-a} f^{(n)}(b) \right],$$

where the following is the case:

$$P_1(b) = \frac{1}{(n-1) \cdot n} \left(\int_a^b g(x)(x-a)^n dx - \frac{\lambda^{n+1}}{n+1} \right)$$
(9)

and the following is obtained.

$$m_1 = a + \frac{1}{(n-1) \cdot n \cdot (n+1) \cdot P_1(b)} \left(\int_a^b g(x)(x-a)^{n+1} dx - \frac{\lambda^{n+2}}{n+2} \right).$$
(10)

Proof. Since $f^{(n-1)}$ is absolutely continuous, function f satisfies the conditions of Theorem 3. Therefore, identity (3) holds.

From condition $0 \le g \le 1$, function G_1 defined by (7) is non-negative. Hence, for every $n \ge 2$, we have the following.

$$\int_{t}^{b} G_{1}(x)(x-t)^{n-2}dx \ge 0, \quad t \in [a,b].$$

Define

$$p(t) = \int_t^b G_1(x)(x-t)^{n-2} dx.$$

Since the function f is (n + 2)-convex, function $f^{(n)}$ is convex. Furthermore, function p is non-negative, so we can apply Theorem 1 and obtain the following inequality:

$$P_{1}(b) \cdot f^{(n)}(m_{1}) \leq \int_{a}^{b} \left(\int_{t}^{b} G_{1}(x)(x-t)^{n-2} dx \right) f^{(n)}(t) dt$$

$$\leq P_{1}(b) \cdot \left[\frac{b-m_{1}}{b-a} f^{(n)}(a) + \frac{m_{1}-a}{b-a} f^{(n)}(b) \right],$$
(11)

where $P_1(b)$ and m_1 are given by

$$P_1(b) = \int_a^b \left(\int_t^b G_1(x)(x-t)^{n-2} dx \right) dt$$

and

$$m_1 = \frac{1}{P_1(b)} \int_a^b \left(\int_t^b G_1(x)(x-t)^{n-2} dx \right) t \, dt$$

By calculating $P_1(b)$ and m_1 , we obtain the following:

$$\begin{split} P_1(b) &= \int_a^b \left(\int_t^b G_1(x)(x-t)^{n-2} dx \right) dt \\ &= \int_a^{a+\lambda} \left(\int_a^x (1-g(s)) ds \right) \frac{(x-a)^{n-1}}{n-1} dx + \int_{a+\lambda}^b \left(\int_x^b g(s) ds \right) \frac{(x-a)^{n-1}}{n-1} dx \\ &= \int_a^{a+\lambda} \frac{(x-a)^n}{n-1} dx + \lambda \cdot \int_{a+\lambda}^b \frac{(x-a)^{n-1}}{n-1} dx - \int_a^b \left(\int_a^x g(s) ds \right) \frac{(x-a)^{n-1}}{n-1} dx \\ &= \frac{-\lambda^{n+1}}{(n-1) \cdot n \cdot (n+1)} + \int_a^b g(x) \frac{(x-a)^n}{(n-1) \cdot n} dx \end{split}$$

and

$$\begin{split} m_1 &= \frac{1}{P_1(b)} \int_a^b \left(\int_t^b G_1(x)(x-t)^{n-2} dx \right) t \, dt \\ &= \frac{1}{P_1(b)} \int_a^b G_1(x) \left(\int_a^x (x-t)^{n-2} \cdot t \, dt \right) dx \\ &= \frac{1}{P_1(b)} \int_a^b G_1(x) \left(t \cdot \frac{-(x-t)^{n-1}}{n-1} \Big|_a^x + \int_a^x \frac{(x-t)^{n-1}}{n-1} dt \right) dx \\ &= \frac{1}{P_1(b)} \int_a^b G_1(x) \left(\frac{a \cdot (x-a)^{n-1}}{n-1} + \frac{(x-a)^n}{(n-1) \cdot n} \right) dx \\ &= a + \frac{1}{P_1(b)} \int_a^b G_1(x) \frac{(x-a)^n}{(n-1) \cdot n} dx \\ &= a + \frac{1}{P_1(b)} \left(\frac{-\lambda^{n+2}}{(n-1) \cdot n \cdot (n+1) \cdot (n+2)} + \int_a^b g(x) \frac{(x-a)^{n+1}}{(n-1) \cdot n \cdot (n+1)} dx \right) \end{split}$$

Using identity (3) for the middle part of the inequality (11), inequality (11) becomes inequality (8). Hence, the proof is completed. \Box

Theorem 6. Let $f: [a,b] \to \mathbb{R}$ be (n+2)-convex on [a,b] and $f^{(n-1)}$ absolutely continuous for $n \ge 2$. Let $g: [a,b] \to \mathbb{R}$ be an integrable function such that $0 \le g \le 1$ and $\lambda = \int_a^b g(t)dt$. Let function G_1 be defined by (7). If the following is the case:

$$\int_{a}^{t} G_{1}(x)(x-t)^{n-2} dx \leq 0, \quad t \in [a,b],$$

then we have the following:

$$P_{2}(b) \cdot f^{(n)}(m_{2}) \leq (n-2)! \left[\int_{a}^{b} f(t)g(t)dt - \int_{a}^{a+\lambda} f(t)dt - \sum_{i=0}^{n-2} \frac{f^{(i+1)}(b)}{i!} \int_{a}^{b} G_{1}(x)(x-b)^{i}dx \right] \leq P_{2}(b) \cdot \left[\frac{b-m_{2}}{b-a} f^{(n)}(a) + \frac{m_{2}-a}{b-a} f^{(n)}(b) \right],$$

$$(12)$$

where

$$P_2(b) = \frac{1}{(n-1) \cdot n} \left(\frac{(a-b)^{n+1} - (a+\lambda-b)^{n+1}}{n+1} + \int_a^b g(x)(x-b)^n dx \right)$$

and

$$m_{2} = b + \frac{1}{(n-1) \cdot n \cdot (n+1) \cdot P_{2}(b)} \\ \times \left(\frac{(a-b)^{n+2} - (a+\lambda-b)^{n+2}}{n+2} + \int_{a}^{b} g(x)(x-b)^{n+1} dx \right)$$

Proof. If we assume the following:

$$\int_{a}^{t} G_{1}(x)(x-t)^{n-2}dx \le 0, \quad t \in [a,b]$$

then we have the following.

$$-\int_{a}^{t} G_{1}(x)(x-t)^{n-2}dx \ge 0, \quad t \in [a,b].$$

Now similarly to the proof of Theorem 5 using the following non-negative function:

$$p(t) = -\int_{a}^{t} G_{1}(x)(x-t)^{n-2}dx$$

and identity (4), we obtain inequality (12). Similarly, we calculate the expressions for $P_2(b)$ and m_2 and obtain the following:

$$\begin{split} P_2(b) &= -\int_a^b \left(\int_a^t G_1(x)(x-t)^{n-2} dx \right) dt \\ &= \int_a^{a+\lambda} \left(\int_a^x (1-g(s)) ds \right) \frac{(x-b)^{n-1}}{n-1} dx + \int_{a+\lambda}^b \left(\int_x^b g(s) ds \right) \frac{(x-b)^{n-1}}{n-1} dx \\ &= \int_a^{a+\lambda} (x-a) \frac{(x-b)^{n-1}}{n-1} dx + \lambda \cdot \int_{a+\lambda}^b \frac{(x-b)^{n-1}}{n-1} dx - \int_a^b \left(\int_a^x g(s) ds \right) \frac{(x-b)^{n-1}}{n-1} dx \\ &= \frac{(a-b)^{n+1}}{(n-1) \cdot n \cdot (n+1)} - \frac{(a+\lambda-b)^{n+1}}{(n-1) \cdot n \cdot (n+1)} + \int_a^b g(x) \frac{(x-b)^n}{(n-1) \cdot n} dx \end{split}$$

and

$$\begin{split} m_2 &= -\frac{1}{P_2(b)} \int_a^b \left(\int_a^t G_1(x)(x-t)^{n-2} dx \right) t \, dt \\ &= -\frac{1}{P_2(b)} \int_a^b G_1(x) \left(\int_x^b (x-t)^{n-2} \cdot t \, dt \right) dx \\ &= -\frac{1}{P_2(b)} \int_a^b G_1(x) \left(t \cdot \frac{-(x-t)^{n-1}}{n-1} \Big|_x^b + \int_x^b \frac{(x-t)^{n-1}}{n-1} dt \right) dx \\ &= -\frac{1}{P_2(b)} \int_a^b G_1(x) \left(-b \cdot \frac{(x-b)^{n-1}}{n-1} - \frac{(x-b)^n}{(n-1) \cdot n} \right) dx \\ &= b + \frac{1}{P_2(b)} \int_a^b G_1(x) \frac{(x-b)^n}{(n-1) \cdot n} dx \\ &= b + \frac{1}{(n-1) \cdot n \cdot (n+1) \cdot P_1(b)} \\ &\times \left(\frac{(a-b)^{n+2}}{n+2} - \frac{(a+\lambda-b)^{n+2}}{n+2} + \int_a^b g(x)(x-b)^{n+1} dx \right). \end{split}$$

Hence, the proof is completed. \Box

Theorem 7. Let $f: [a, b] \to \mathbb{R}$ be (n + 2)-convex on [a, b] and $f^{(n-1)}$ absolutely continuous for $n \ge 2$. Let $g: [a, b] \to \mathbb{R}$ be an integrable function such that $0 \le g \le 1$ and $\lambda = \int_a^b g(t) dt$. Let function G_2 be defined by the following.

$$G_{2}(x) = \begin{cases} \int_{a}^{x} g(t)dt, & x \in [a, b - \lambda], \\ \int_{x}^{b} (1 - g(t))dt, & x \in [b - \lambda, b]. \end{cases}$$
(13)

Then, the following is obtained:

$$P_{3}(b) \cdot f^{(n)}(m_{3}) \leq (n-2)! \left[\int_{b-\lambda}^{b} f(t)dt - \sum_{i=0}^{n-2} \frac{f^{(i+1)}(a)}{i!} \int_{a}^{b} G_{2}(x)(x-a)^{i}dx - \int_{a}^{b} f(t)g(t)dt \right]$$
(14)
$$\leq P_{3}(b) \cdot \left[\frac{b-m_{3}}{b-a} f^{(n)}(a) + \frac{m_{3}-a}{b-a} f^{(n)}(b) \right],$$

where

$$P_3(b) = \frac{1}{(n-1) \cdot n} \left(\frac{(b-a)^{n+1} - (b-\lambda-a)^{n+1}}{n+1} - \int_a^b g(x)(x-a)^n dx \right)$$

and

$$m_{3} = a + \frac{1}{(n-1) \cdot n \cdot (n+1) \cdot P_{3}(b)} \\ \times \left(\frac{(b-a)^{n+2} - (b-\lambda-a)^{n+2}}{n+2} - \int_{a}^{b} g(x)(x-a)^{n+1} dx\right).$$

Proof. We follow the similar arguments as in the proof of Theorem 5. As function $f^{(n-1)}$ is absolutely continuous, the identity (5) holds. The inequality (14) follows directly from Theorem 1, substituting the non-negative function p by a non-negative function of the following:

$$p(t) = \int_t^b G_2(x)(x-t)^{n-2} dx$$

and a convex function f by a convex function $f^{(n)}$, and then using identity (5) for integral $\int_a^b \left(\int_t^b G_2(x)(x-t)^{n-2}dx\right)f^{(n)}(t)dt$. Furthermore, we calculate $P_3(b)$ and m_3 as follows.

$$\begin{split} P_{3}(b) &= \int_{a}^{b} \left(\int_{t}^{b} G_{2}(x)(x-t)^{n-2} dx \right) dt \\ &= \int_{a}^{b-\lambda} \left(\int_{a}^{x} g(s) ds \right) \frac{(x-a)^{n-1}}{n-1} dx + \int_{b-\lambda}^{b} \left(\int_{x}^{b} (1-g(s)) ds \right) \frac{(x-a)^{n-1}}{n-1} dx \\ &= \int_{b-\lambda}^{b} (b-x) \frac{(x-a)^{n-1}}{n-1} dx - \lambda \cdot \int_{b-\lambda}^{b} \frac{(x-a)^{n-1}}{n-1} dx + \int_{a}^{b} \left(\int_{a}^{x} g(s) ds \right) \frac{(x-a)^{n-1}}{n-1} dx \\ &= \frac{(b-a)^{n+1} - (b-\lambda-a)^{n+1}}{(n-1) \cdot n \cdot (n+1)} - \int_{a}^{b} g(x) \frac{(x-a)^{n}}{(n-1) \cdot n} dx, \end{split}$$

$$\begin{split} m_3 &= \frac{1}{P_3(b)} \int_a^b \left(\int_t^b G_2(x)(x-t)^{n-2} dx \right) t \, dt \\ &= \frac{1}{P_3(b)} \int_a^b G_2(x) \left(\int_a^x (x-t)^{n-2} \cdot t \, dt \right) dx \\ &= \frac{1}{P_3(b)} \int_a^b G_2(x) \left(t \cdot \frac{-(x-t)^{n-1}}{n-1} \Big|_a^x + \int_a^x \frac{(x-t)^{n-1}}{n-1} dt \right) dx \\ &= \frac{1}{P_3(b)} \int_a^b G_2(x) \left(\frac{a \cdot (x-a)^{n-1}}{n-1} + \frac{(x-a)^n}{(n-1) \cdot n} \right) dx \\ &= a + \frac{1}{P_3(b)} \int_a^b G_2(x) \frac{(x-a)^n}{(n-1) \cdot n} dx \\ &= a + \frac{1}{P_3(b)} \left(\frac{(b-a)^{n+2} - (b-\lambda-a)^{n+2}}{(n-1) \cdot n \cdot (n+1) \cdot (n+2)} - \int_a^b g(x) \frac{(x-a)^{n+1}}{(n-1) \cdot n \cdot (n+1)} dx \right). \end{split}$$

Hence, the proof is completed. \Box

Theorem 8. Let $f: [a, b] \to \mathbb{R}$ be (n + 2)-convex on [a, b] and $f^{(n-1)}$ absolutely continuous for $n \ge 2$. Let $g: [a, b] \to \mathbb{R}$ be an integrable function such that $0 \le g \le 1$ and $\lambda = \int_a^b g(t) dt$. Let function G_2 be defined by (13). If the following is the case:

$$\int_{a}^{t} G_{2}(x)(x-t)^{n-2} dx \le 0, \quad t \in [a,b]$$

then we obtain the following:

$$P_{4}(b) \cdot f^{(n)}(m_{4}) \leq (n-2)! \left[\int_{b-\lambda}^{b} f(t)dt - \sum_{i=0}^{n-2} \frac{f^{(i+1)}(b)}{i!} \int_{a}^{b} G_{2}(x)(x-b)^{i}dx - \int_{a}^{b} f(t)g(t)dt \right]$$
(15)
$$\leq P_{4}(b) \cdot \left[\frac{b-m_{4}}{b-a} f^{(n)}(a) + \frac{m_{4}-a}{b-a} f^{(n)}(b) \right],$$

where

$$P_4(b) = \frac{-1}{(n-1)\cdot n} \left(\frac{(-\lambda)^{n+1}}{n+1} + \int_a^b g(x)(x-b)^n dx \right)$$

and

$$m_4 = b - \frac{1}{(n-1) \cdot n \cdot (n+1) \cdot P_4(b)} \left(\frac{(-\lambda)^{n+2}}{n+2} + \int_a^b g(x)(x-b)^{n+1} dx \right).$$

Proof. Under the assumption that $\int_a^t G_2(x)(x-t)^{n-2} dx \le 0$, it is obvious that the following is the case:

$$p(t) = -\int_{a}^{t} G_{2}(x)(x-t)^{n-2}dx$$
(16)

where it is a non-negative function. Again, replacing p(t) in Theorem 1 by (16) and f by $f^{(n)}$ and then using the identity (6) for

$$\int_a^b \left(\int_a^t G_2(x)(x-t)^{n-2}dx\right) f^{(n)}(t)dt,$$

we obtain the required inequalities (15). Finally, a simple calculation yields the following:

$$\begin{split} P_4(b) &= -\int_a^b \left(\int_a^t G_2(x)(x-t)^{n-2} dx \right) dt \\ &= \int_a^{b-\lambda} \left(\int_a^x g(s) ds \right) \frac{(x-b)^{n-1}}{n-1} dx + \int_{b-\lambda}^b \left(\int_x^b (1-g(s)) ds \right) \frac{(x-b)^{n-1}}{n-1} dx \\ &= -\int_{b-\lambda}^b \frac{(x-b)^n}{n-1} dx - \lambda \cdot \int_{b-\lambda}^b \frac{(x-b)^{n-1}}{n-1} dx + \int_a^b \left(\int_a^x g(s) ds \right) \frac{(x-b)^{n-1}}{n-1} dx \\ &= -\frac{(-\lambda)^{n+1}}{(n-1) \cdot n \cdot (n+1)} - \int_a^b g(x) \frac{(x-b)^n}{(n-1) \cdot n} dx \end{split}$$

and

$$\begin{split} m_4 &= \frac{-1}{P_4(b)} \int_a^b \left(\int_a^t G_2(x)(x-t)^{n-2} dx \right) t \, dt \\ &= \frac{-1}{P_4(b)} \int_a^b G_2(x) \left(\int_x^b (x-t)^{n-2} \cdot t \, dt \right) dx \\ &= \frac{-1}{P_4(b)} \int_a^b G_2(x) \left(t \cdot \frac{-(x-t)^{n-1}}{n-1} \Big|_x^b + \int_x^b \frac{(x-t)^{n-1}}{n-1} dt \right) dx \\ &= \frac{-1}{P_4(b)} \int_a^b G_2(x) \left(-b \cdot \frac{(x-b)^{n-1}}{n-1} - \frac{(x-b)^n}{(n-1) \cdot n} \right) dx \\ &= b + \frac{1}{P_4(b)} \int_a^b G_2(x) \frac{(x-b)^n}{(n-1) \cdot n} dx \\ &= b - \frac{1}{P_4(b)} \left(\frac{(-\lambda)^{n+2}}{(n-1) \cdot n \cdot (n+1) \cdot (n+2)} + \int_a^b g(x) \frac{(x-b)^{n+1}}{(n-1) \cdot n \cdot (n+1)} dx \right). \end{split}$$

Remark 1. If function f is (n + 2)-concave, the inequalities in Theorems 5–8 are reversed. This follows from the fact that for (n + 2)-concave function, we have $-f^{(n+2)} \ge 0$. Hence, $-f^{(n)}$ is convex and we can apply inequality (1) to function $-f^{(n)}$.

Remark 2. The expressions $P_i(b)$ and m_i for i = 1, ..., 4 can also be achieved by the method introduced in [16]. By this method, we calculate $P_1(b)$ and m_1 . Other expressions can be recaptured in a similar manner.

The value of $P_1(b)$ can be obtained from (3) by taking $f(t) = \frac{(t-a)^n}{n!}$. Then, $f^{(n)}(t) = 1$. Thus, we have the following.

$$P_{1}(b) = -(n-2)! \left(\int_{a}^{a+\lambda} \frac{(x-a)^{n}}{n!} dt - \int_{a}^{b} \frac{(x-a)^{n}}{n!} g(t) dt \right)$$
$$= -\frac{\lambda^{n+1}}{(n-1) \cdot n \cdot (n+1)} + \int_{a}^{b} \frac{(x-a)^{n}}{(n-1) \cdot n} g(t) dt.$$

Hence, we obtained expression (9).

From Theorem 1, we previously obtained the following.

$$m_1 = \frac{1}{P_1(b)} \int_a^b \left(\int_t^b G_1(x)(x-t)^{n-2} dx \right) t \, dt.$$

To calculate m_1 , we take function $f(t) = \frac{(t-a)^{n+1}}{(n+1)!}$. Then, $f^{(n)}(t) = t - a$. Hence, from the identity (3), we obtain expression (10).

3. Conclusions

In this paper, we obtained new weighted Hermite–Hadamard-type inequalities for higher order convex functions. We used previously obtained identities related to the generalizations of Steffensen's inequality. Results obtained in this paper can be considered as a starting point for some future work.

Author Contributions: Conceptualization, J.P., A.P.P. and K.S.K.; Writing — original draft, J.P., A.P.P. and K.S.K. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: Croatian Science Foundation (HRZZ 7926) "Separation of parameter influence in engineering modeling and parameter identification".

Conflicts of Interest: The authors declare no conflict of interest.

References

- 1. Hermite, C. Sur deux limites d'une intégrale dé finie. *Mathesis* 1883, 3, 82.
- 2. Hadamard, J. Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann. *J. Math. Pures Appl.* **1893**, *58*, 171–215.
- 3. Dinu, C. A weighted Hermite Hadamard inequality for Steffensen-Popoviciu and Hermite-Hadamard weights on time scales, Analele Stiintifice ale Universitatii Ovidius Constanta–Seria. *Matematica* **2009**, *17*, 77–90.
- Abbas, M.I.; Ragusa, M.A. Solvability of Langevin equations with two Hadamard fractional derivatives via Mittag-Leffler functions. *Appl. Anal.* 2021, 1–15. [CrossRef]
- 5. Niculescu, C.P.; Persson, L.-E. Old and new on the Hermite-Hadamard inequality. Real Anal. Exch. 2004, 29, 663–685. [CrossRef]
- 6. Niculescu, C.P.; Persson, L.-E. *Convex Functions and Their Applications: A Contemporary Approach*; CMS Books in Mathematics; Springer: New York, NY, USA, 2005.
- Niculescu, C.P.; Stanescu, M.M. The Steffensen-Popoviciu measures in the context of qualiconvex functions. *J. Math. Inequal.* 2017, 11, 469–483. [CrossRef]
- 8. Pečarić, J.E.; Proschan, F.; Tong, Y.L. Convex functions, partial orderings, and statistical applications. In *Mathematics in Science and Engineering* 187; Academic Press: San Diego, CA, USA, 1992.
- Pečarić, J.; Perić, I. Refinements of the integral form of Jensen's and Lah-Ribarič inequalities and applications for Csiszár divergence. J. Inequal. Appl. 2020, 108, 287 [CrossRef]
- 10. Wu, S. On the Weighted Generalization of the Hermite-Hadamard Inequality and Its Applications. *Rocky Mountain J. Math.* 2009, 39, 1741–1749. [CrossRef]
- 11. Steffensen , J.F. On certain inequalities between mean values and their application to actuarial problems. *Skand. Actuar. J.* **1918**, 1918, 82–97. [CrossRef]
- 12. Jakšetić, J.; Pečarić, J.; Perušić Pribanić, A.; Smoljak Kalamir, K. Weighted Steffensen's Inequality (Recent Advances in Generalizations of Steffensen's Inequality); Monograhps Inequalities 17: Element, Zagreb, 2020.
- Pečarić, J.; Smoljak Kalamir, K.; Varošanec, S. Steffensen's and Related Inequalities (A Comprehensive Survey and Recent Advances). Monograhps in Inequalities 7. 2014. Available online: http://ele-math.com/static/pdf/books/593-mia07.pdf (accessed on 29 March 2022).
- Mitrinović, D.S. *The Steffensen Inequality*; Publikacije Elektrotehničkog Fakulteta. Serija Matematika i Fizika No. 247/273; University of Belgrade: Belgrade, Serbia, 1969; pp. 1–14.
- 15. Pečarić, J.; Perušić Pribanić, A.; Smoljak Kalamir, K. Generalizations of Steffensen's inequality via Taylor's formula. *J. Inequal. Appl.* **2015**, 207, 1–25. [CrossRef]
- 16. Barić, J.; Kvesić, L.J.; Pečarić, J.; Ribičić Penava, M. Fejér type inequalities for higher order convex functions and quadrature formulae. *Aequat. Math.* **2021**, *96*, 417–430. [CrossRef]