Analytical Solution for the Problem of Pure Bending of Orthotropic Micropolar Plate

Jurković, Damjan; Jelenić, Gordan; Grbčić Erdelj, Sara

Original scientific paper UDK: 539.3
https://doi.org/10.32762/zr.27.1.8

ANALYTICAL SOLUTION FOR THE PROBLEM OF PURE BENDING OF ORTHOTROPIC MICROPOLAR PLATE

ANALITIČKO RJEŠENJE PROBLEMA ČISTOG SAVIJANJA ORTOTROPNE MIKROPOLARNE PLOČE

Damjan Jurković [*], Gordan Jelenić [*], Sara Grbčić Erdelj [*]

Abstract

When analysing the problem of pure bending of a plate, it can be noticed that, for cylindrical bending, a spatial problem collapses into a plane-strain problem. For such a boundary-value problem of the Cosserats' continuum, three engineering parameters are required: Young's modulus, Poisson's ratio and the characteristic length for bending. Here we consider an orthotropic form of such a problem, whereby two Young's moduli, four Poison's ratios and one characteristic length for bending are found to be sufficient to propose a mathematical model of this problem. General equations of the isotropic micropolar continuum are introduced, and the analytical solution for the pure bending of an isotropic micropolar plate is generalized to the case of orthotropic microstructure. By defining the ratio of a distributed force and a distributed moment boundary conditions required for the pure-bending state, a closed-form solution to this problem is obtained in terms of displacement, strains and stress functions. It is shown that the derived results reduce to the isotropic ones if a material isotropy is assumed.

Key words: *orthotropic micropolar continuum, analytical solution of pure bending, characteristic length for bending*

Sažetak

Pri analizi problema čistog savijanja ploče moguće je uočiti kako se, za cilindrično savijanje, prostorni problem može svesti na problem ravninskog stanja deformacija. Za takav problem rubnih vrijednosti Cosseratovog kontinuuma potrebna su tri materijalna parametra: Youngov modul elastičnosti, Poissonov koeficijent i

* Sveučilište u Rijeci, Građevinski fakultet, Radmile Matejčić 3, 51000 Rijeka
E-mail: {damjan.jurkovic, gordan.jelenic, sara.grbcic}@gradri.uniri.hr

Rad zaprimljen: 9.6.2024. Rad prihvaćen: 26.9.2024.

karakteristična duljina za savijanje. Ovdje razmatramo ortotropni oblik takvog problema, gdje su dva Youngova modula, četiri Poissonova koeficijenta i jedna karakteristična duljina za savijanje dovoljni za definiranje matematičkog modela problema. Predstavljene su opće jednadžbe izotropnog mikropolarnog kontinuuma, te je analitičko rješenje za problem čistog savijanja mikropolarne ploče generalizirano na slučaj ortotropne mikrostrukture. Definiranjem omjera rubnih uvjeta distribuiranih sila i distribuiranih momenata potrebnih za stanje čistog savijanja dobiveno je rješenje u zatvorenom obliku, izraženo preko funkcija pomaka, deformacija i naprezanja. Prikazano je da se uvođenjem materijalne izotropije izvedeni izrazi svode na poznate izotropne rezultate.

Ključne riječi: *ortotropni mikropolarni kontinuum, analitičko rješenje čistog savijanja, karakteristična duljina za savijanje*

1. Introduction

Micropolar elasticity theory, also known as Cosserats' theory, is a generalised continuum theory. As the theory uses an additional kinematic field, the so-called *microrotation filed*, six engineering parameters are needed, compared to just two used in classical elasticity. Identification of these parameters is the subject of numerous studies.

This paper is setting theoretical foundation for identification of one of those micropolar parameters, the so-called *characteristic length for bending*, for an orthotropic micropolar continuum. In an effort to obtain this parameter, Gauthier's closed-form analytical solution for the pure bending of an isotropic micropolar plate [1] is generalised for the case of orthotropic micropolar material. It should be noted that the present orthotropic micropolar continuum does not account for all the possible effects outlined in [2], which would demand a total of 24 material parameters. Rather, orthotropy is introduced just in the stress-strain constitutive tensor, while the constitutive tensor relating the couple-stresses and the curvatures is assumed to remain isotropic. This results in just one characteristic length for bending for a planar problem. Furthermore, the problem of pure bending does not demand any other micropolar parameters, which leaves us with seven parameters in total.

The overall notation used for the material parameters is as in [3], and a summation convention is used throughout this work.

2. Governing equations of the micropolar elasticity theory

Compared to the classical Cauchy continuum, micropolar continuum is described by six degrees of freedom, three translational and three rotational. The rotational degrees of freedom (microrotations) are independent from the rotations resulting from the displacements as the skew-symmetric part of the displacement tensor (the so-called macrorotations).

Let us consider a homogeneous isotropic body of volume V bounded by a surface S, subject to volume force and moment loads p_V and m_V and surface force and moment loads p_s and m_s. The deformation of the body is defined by the displacement vector $\mathbf{u}$ and the microrotation vector $\boldsymbol{\varphi}$, while the stress state is defined by the stress tensor $\boldsymbol{\sigma}_{ij}$ and the couple-stress tensor $\boldsymbol{\mu}_{ij}$. These stresses are connected to the strain tensor $\boldsymbol{\epsilon}_{ij}$ and the curvature tensor $\boldsymbol{\kappa}_{ij}$. All four of these tensors are non-symmetric. The force equilibrium equations can be written as

$$\sigma_{ij,j} + p_{Vi} = 0, \qquad (1)$$

where *i, j = x, y, z.* The first index to the stress tensor denotes the coordinate direction of stress action, and the second index direction of the surface normal. The comma denotes a spatial differentiation in a given direction. From the moment equilibrium, the second set of equilibrium equations is derived as

$$\mu_{ij,j} - \varepsilon_{ijk}\sigma_{jk} + m_{Vi} = 0, \qquad (2)$$

where ε_{ijk} is the Levi-Civita permutation tensor. In the couple-stress tensor the first index again denotes the axial direction of the couple-stress action and the second index denotes the surface normal. Components of stresses and couple-stresses acting on an infinitesimal two-dimensional body are shown in Figure 1. The stress tensor is non-symmetric, except for the case when $\mu_{ij} = 0$ and $m_{Vi} = 0$, which is equivalent to the classical elasticity theory.

The force and moment equilibria on the body surface read

$$\sigma_{ij}n_j = p_{si}, \qquad (3)$$

$$\mu_{ij}n_j = m_{si}, \qquad (4)$$

where n_j is the surface normal vector. Equations (3) and (4) constitute the so-called *natural* or *load* boundary conditions, while the so-called *essential* or *geometric* boundary conditions (on $\mathbf{u}$ and $\boldsymbol{\varphi}$) are applied simply by prescribing the actual displacements and microrotations to the boundary.

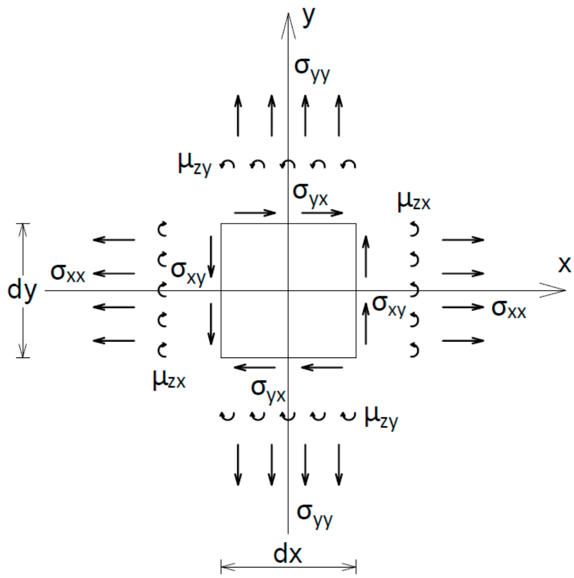

Figure 1. Stress and couple-stress components acting on a two-dimensional body

The kinematic equations are defined by

$$\epsilon_{ij} = u_{i,j} + \varepsilon_{ijk}\,\varphi_k, \tag{5}$$

$$\kappa_{ij} = \varphi_{i,j}. \tag{6}$$

from which can be seen that the strain and curvature tensors are also non-symmetric. The constitutive equations read

$$\sigma_{ij} = T_{ijpq}\epsilon_{pq}, \tag{7}$$

$$\mu_{ij} = D_{ijpq}\kappa_{pq}, \tag{8}$$

where *i, j, p, q = x, y, z.* For an isotropic continuum, these equations may be written as

$$\sigma_{ij} = \lambda\epsilon_{pp}\delta_{ij} + (\mu+\nu)\epsilon_{ij} + (\mu-\nu)\epsilon_{ji}, \tag{9}$$

$$\mu_{ij} = \alpha\kappa_{pp}\delta_{ij} + (\beta+\gamma)\kappa_{ij} + (\beta-\gamma)\kappa_{ji}, \tag{10}$$

where δ_{ij} is the Kronecker delta symbol. Here μ and λ are known from classical elasticity as the Lamé constants, while ν, α, β, γ are additional micropolar material constants. These material constants can be described by six engineering constants: Young's modulus E, Poisson's ratio n, coupling number between the microrotation and microrotation $0 < N < 1$, polar ratio $0 \leq \psi \leq 3/2$, and the characteristic lengths for torsion and bending, l_t and l_b respectively. The relationships between the two sets of parameters can be found in [4].

Since kinematic microstrain system (5) is overdetermined, displacement and microrotation fields cannot be defined uniquely. To ensure uniqueness and continuity of the kinematic fields it is necessary to apply constraints to strain values ϵ_{ij} and κ_{ij} via compatibility conditions:

$$\begin{gathered} \epsilon_{ik,j} + \varepsilon_{kil}\kappa_{lj} = \epsilon_{jk,i} + \varepsilon_{kjl}\kappa_{li} \\ \kappa_{kj,i} = \kappa_{ki,j}\,, \end{gathered} \tag{11}$$

with $i, j, k, l = x, y, z$. Let us next look at the planar problem in xy-plane. For plane problems, the kinematic fields involve only three components: displacements u_x and u_y and microrotation φ_z. This is sometimes called the first planar problem (e.g. [5]). Next, if we assume that the planar problem is that of a plane-strain type, the remaining stresses are $\boldsymbol{\sigma} = \langle \sigma_{xx} \quad \sigma_{yy} \quad \sigma_{xy} \quad \sigma_{yx} \rangle^T$ and $\boldsymbol{\mu} = \langle \mu_{zx} \quad \mu_{zy} \rangle^T$, and the remaining strains $\boldsymbol{\epsilon} = \langle \epsilon_{xx} \quad \epsilon_{yy} \quad \epsilon_{xy} \quad \epsilon_{yx} \rangle^T$ and $\boldsymbol{\kappa} = \langle \kappa_{zx} \quad \kappa_{zy} \rangle^T$, while σ_{zz} ceases to be an independent stress component. It is shown in [3] that in this case the material parameter α disappears. Finally, compatibility conditions are simplified to just three equations and read

$$\begin{aligned} \frac{\partial \epsilon_{xx}}{\partial y} + \kappa_{zx} &= \frac{\partial \epsilon_{xy}}{\partial x}, \\ \frac{\partial \epsilon_{yy}}{\partial x} - \kappa_{zy} &= \frac{\partial \epsilon_{yx}}{\partial y}, \\ \frac{\partial \kappa_{zy}}{\partial x} &= \frac{\partial \kappa_{zx}}{\partial y}. \end{aligned} \tag{13}$$

3. Reduced orthotropic material model

Orthotropic materials are the materials whose material properties differ in the directions of a set of orthogonal axes. Considering classical theory, these materials have three Young's moduli E_i and six Poisson's ratios n_{ij}, where $i, j = x, y, z$ and $i \neq j$. The first index next to a Poisson's ratio denotes the direction of the applied axial strain, and the second index denotes the direction of the resulting lateral strain. We will next introduce orthotropy to our material model.

Micropolar orthotropic plane strain constitutive equations are defined in [6] as

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \\ \sigma_{yx} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & 0 \\ A_{12} & A_{22} & 0 & 0 \\ 0 & 0 & A_{77} & A_{78} \\ 0 & 0 & A_{78} & A_{88} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \\ \epsilon_{yx} \end{Bmatrix}, \tag{14}$$

where A_{ij} are undefined material parameters. According to [7], the constitutive equation for the normal strains reads

$$\begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_x} & -\frac{n_{yx}}{E_y} & -\frac{n_{zx}}{E_z} \\ -\frac{n_{xy}}{E_x} & \frac{1}{E_y} & -\frac{n_{zy}}{E_z} \\ -\frac{n_{xz}}{E_x} & -\frac{n_{yz}}{E_y} & \frac{1}{E_z} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix}. \tag{15}$$

Since the compliance matrix in (15) is symmetric, the number of independent engineering parameters is reduced to six. It may be shown that for the plane strain problem this equation can be rewritten as

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \end{Bmatrix} = \frac{\tilde{E}}{1-\alpha_o^2} \begin{bmatrix} \psi_o^{-1} & \alpha_o \\ \alpha_o & \psi_o \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \end{Bmatrix}, \tag{16}$$

where $\tilde{E}$, α_o, ψ_o are introduced to relate the material parameters as follows:

$$\tilde{E} = \sqrt{\tilde{E}_x \tilde{E}_y},$$

$$\psi_o = \sqrt{\frac{\tilde{E}_y}{\tilde{E}_x}},$$

$$\alpha_o = \frac{\tilde{E}}{E_y}\left(n_{yx} + n_{yz} n_{zx}\right) = \frac{\tilde{E}}{E_x}\left(n_{xy} + n_{zy} n_{xz}\right), \tag{17}$$

$$\tilde{E}_x = \frac{E_x}{1 - n_{zx} n_{xz}},$$

$$\tilde{E}_y = \frac{E_y}{1 - n_{yz} n_{zy}}.$$

Parameters A_{11}, A_{12} and A_{22} in (14) can be obtained from (16). In this paper parameters A_{77}, A_{78} and A_{88} can remain undefined as they will be proven to vanish in further analysis. As already mentioned, in the present model, the constitutive tensor relating couple stresses to curvatures is understood to keep its isotropic form as in (10).

4. Pure bending of a micropolar plate

Let us observe a three-dimensional micropolar isotropic plate with a thickness h, length L and width B, where $L, B >> h$. The plate is subjected to distributed edge moments M_x and M_z. It is shown in [8] that by setting $M_z = nM_x$ bending of a plate is no longer anticlastic. Now the curvature in yz-plane disappears and the plate bends in a cylindrical shape, hence the problem is reduced to that of a plane strain type. From here we assume that the isotropic plane strain problem is just as described in the last section before the conclusions of this paper, where we show that the orthotropic solution we are set to derive includes this isotropic solution as a special case. Figure 2 shows bending happening in xy-plane, where now, for a planar case $M = BM_x$.

Bending moment M may take place as a result of a linearly changing continuous load p_{sx} applied to the boundary:

$$p_{sx} = -\frac{2p_0}{h}y. \tag{18}$$

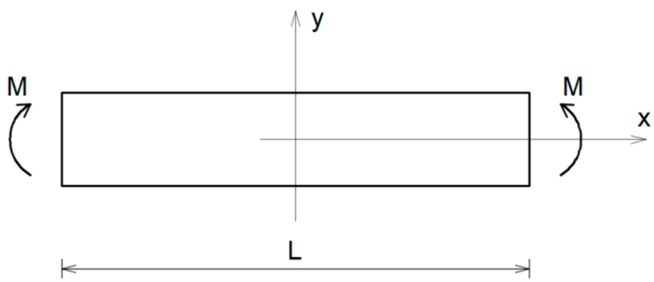

Figure 2. Cross-sectional view of the pure bending of a plate

While this holds true for the classical elasticity theory, micropolar elasticity requires additional constant continuous couple-stress m_{sz}, as shown in Figure 3. Now the resultant bending moment M is obtained as

$$M = b\int_{-\frac{h}{2}}^{\frac{h}{2}}(-yp_{sx} + m_{sz})\mathrm{d}y = \frac{bh^2}{6}p_0 + bhm_{sz}, \tag{19}$$

from which can be seen that neither p_{sx} nor m_{sz} can be derived uniquely from the applied bending moment. Upon further inspection of boundary conditions it can be noted that all the existing stress components are constant in all directions except for the y direction. Following from that it can be noted from the equilibrium equations (1) and (2), while presuming that the volume forces are equal to zero, that:

$$\frac{\partial \sigma_{xy}}{\partial y} = 0 \implies \sigma_{xy} = \text{const.},$$

$$\frac{\partial \sigma_{yy}}{\partial y} = 0 \implies \sigma_{yy} = \text{const.}, \tag{20}$$

$$\frac{\partial \mu_{zy}}{\partial y} + \sigma_{yx} - \sigma_{xy} = 0 \implies \mu_{zy} = \int (\sigma_{xy} - \sigma_{yx})\mathrm{d}y + \text{const.}.$$

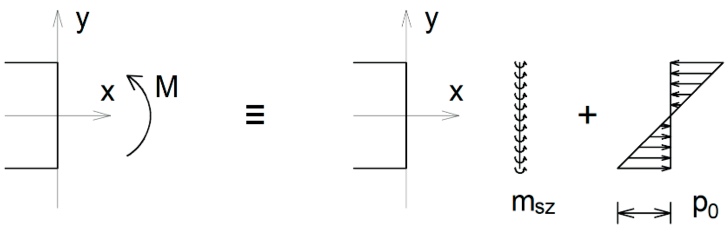

Figure 3. Boundary conditions for micropolar continuum

Since upper and lower boundary do not have any loads applied to them, σ_{xy}, σ_{yy} and μ_{zy} are equal to zero on these edges. Hence, according to (20), σ_{xy} and σ_{yy} disappear and

$$\mu_{zy} = -\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{yx}\mathrm{d}y. \tag{21}$$

For the case of pure bending, the shear strains need to be equal to zero:

$$\epsilon_{xy} = \epsilon_{yx} = 0. \tag{22}$$

From (14) and (22) it is obvious that $\sigma_{yx} = 0$, and then, from (21) $\mu_{zy} = 0$. The only remaining nonzero stresses are thus σ_{xx} and μ_{zx}.

Since $\sigma_{yy} = 0$, constitutive equation (16) reads

$$\epsilon_{yy} = -\frac{\alpha_o}{\psi_o}\epsilon_{xx}. \tag{23}$$

Now constitutive equation (16) for σ_{xx} reads

$$\sigma_{xx} = \frac{\tilde{E}}{\psi_o}\epsilon_{xx}. \tag{24}$$

Constitutive equation (10) for μ_{zy} in plane strain takes the following form:

$$\mu_{zy} = (\beta + \gamma)\kappa_{zy}.$$

But since $\mu_{zy} = 0$ it can be seen from (6) that:

$$\kappa_{zy} = \frac{\partial \varphi_z}{\partial y} = 0$$

which means that φ_z is a function of x only.

The two remaining stresses are defined by constitutive equations (24) and, for the plane strain form of (10):

$$\mu_{zx} = (\beta + \gamma)\kappa_{zx}. \tag{25}$$

Boundary condition (3), after substituting (18) reads

$$\sigma_{xx} = p_{sx} = -\frac{2p_0}{h}y, \tag{26}$$

and (4) reads

$$\mu_{zx} = m_{sz}. \tag{27}$$

By manipulating (24) and substituting (26) we get:

$$\epsilon_{xx} = -2\frac{\psi_o}{\tilde{E}}\frac{p_0}{h}y, \tag{28}$$

and then by substituting (28) into (23):

$$\epsilon_{yy} = 2\frac{\alpha_o}{\tilde{E}}\frac{p_0}{h}y. \tag{29}$$

Finally, by substituting (27) into (25) we obtain

$$\kappa_{zx} = \frac{m_{sz}}{(\beta+\gamma)}. \tag{30}$$

All the strain components are now defined by the applied loads. It can now be noted from the kinematic equations (5) and (6) that

$$\epsilon_{xx} = \frac{\partial u_x}{\partial x} = -2\frac{\psi_o}{\tilde{E}}\frac{p_0}{h}y, \tag{31}$$

$$\epsilon_{yy} = \frac{\partial u_y}{\partial y} = 2\frac{\alpha_o}{\tilde{E}}\frac{p_0}{h}y, \tag{32}$$

$$\kappa_{zx} = \frac{\partial \varphi_z}{\partial x} = \frac{m_{sz}}{\beta+\gamma}. \tag{33}$$

Integration of (33) immediately yields

$$\varphi_z = \frac{m_{sz}}{(\beta+\gamma)}x. \tag{34}$$

By substituting all zero strains into (5) it can be seen that:

$$\frac{\partial u_x}{\partial y} = -\varphi_z, \tag{35}$$

$$\frac{\partial u_y}{\partial x} = \varphi_z. \tag{36}$$

Before we integrate these equations to obtain kinematic field from

$$\mathrm{d}u_x = \frac{\partial u_x}{\partial x}\mathrm{d}x + \frac{\partial u_x}{\partial y}\mathrm{d}y,$$

$$\mathrm{d}u_y = \frac{\partial u_y}{\partial x}\mathrm{d}x + \frac{\partial u_y}{\partial y}\mathrm{d}y,$$

it is necessary to ensure its continuity. We can do so using the compatibility equations (13). Substituting all the values into the first compatibility equation reads

$$-2\frac{\psi_o}{\tilde{E}}\frac{p_0}{h} + \frac{m_{sz}}{\beta+\gamma} = 0, \tag{37}$$

while the second and the third compatibility equations are identically satisfied. Thus, if we want the assumed stress state (constant in x direction and changing in y direction with zero shear stresses – the state of pure bending) to be true, the applied loads p_0 and m_{sz} *must be mutually dependent*. This dependence can be defined from (37) as:

$$\frac{m_{sz}}{p_0} = 2\frac{\psi_o}{\tilde{E}}\frac{1}{h}(\beta+\gamma). \tag{38}$$

It is known from [5] that:

$$\beta+\gamma = 4\mu l_b^2. \tag{39}$$

The Lamé constant μ (the shear modulus) is defined as [8]:

$$\mu = \frac{E}{2(1+n)}. \tag{40}$$

This parameter can be transformed into analogue orthotropic parameter as

$$\mu \equiv \mu_o = \frac{\tilde{E}}{2(1+\alpha_o)}. \tag{41}$$

Substitution of (39) and (41) in (38) results in

$$\frac{m_{sz}}{p_0} = 4\frac{l_b^2}{h}\frac{\psi_o}{(1+\alpha_o)} = \frac{h}{6}\frac{\psi_o}{(1+\alpha_o)}\delta, \tag{42}$$

where

$$\delta = 24\left(\frac{l_b}{h}\right)^2. \tag{43}$$

Now that the dependence between p_0 and m_{sz} has been established, it is possible to calculate the resultant boundary moment M (19) in terms of only one of the loads, i.e.

$$M = \frac{1+\alpha_o+\psi_o\delta}{1+\alpha_o}\frac{bh^2}{6}p_0 = \frac{1+\alpha_o+\psi_o\delta}{\psi_o\delta}bhm_{sz}. \tag{44}$$

The applied loads can now be written in terms of the moment as

$$p_0 = \frac{1+\alpha_o}{1+\alpha_o+\psi_o\delta}\frac{M}{W_z}, \tag{45}$$

$$m_{sz} = \frac{\psi_o\delta}{1+\alpha_o+\psi_o\delta}\frac{M}{A}. \tag{46}$$

By substituting (45) and (46) in (26) and (27) respectively, the solutions for the non-zero stress components are obtained as

$$\sigma_{xx} = -\frac{1+\alpha_o}{1+\alpha_o+\psi_o\delta}\frac{M}{I_z}y, \tag{47}$$

$$\mu_{zx} = \frac{\psi_o\delta}{1+\alpha_o+\psi_o\delta}\frac{M}{A}. \tag{48}$$

By substituting (45) in (28), (29) and substituting (46) in (30), with substitution of (39) and (41) in the final result, the analytical solutions for the strains are obtained as:

$$\epsilon_{xx} = -\frac{(1+\alpha_o)\psi_o}{1+\alpha_o+\psi_o\delta}\frac{M}{\tilde{E}I_z}y, \tag{49}$$

$$\epsilon_{yy} = \frac{(1+\alpha_o)\alpha_o}{1+\alpha_o+\psi_o\delta}\frac{M}{\tilde{E}I_z}y, \tag{50}$$

$$\kappa_{zx} = \frac{(1+\alpha_o)\psi_o}{1+\alpha_o+\psi_o\delta}\frac{M}{\tilde{E}I_z}. \tag{51}$$

From (33), by integrating the curvature in (51) over x we get the microrotation field as

$$\varphi_z = \frac{(1+\alpha_o)\psi_o}{1+\alpha_o+\psi_o\delta}\frac{M}{\tilde{E}I_z}x, \tag{52}$$

which proves that φ_z is a function of *x* only. The displacement fields u_x and u_y must satisfy (31), (32), (35) and (36). The functions that satisfy these conditions are

$$u_x = -\frac{(1+\alpha_o)\psi_o}{1+\alpha_o+\psi_o\delta}\frac{M}{\tilde{E}I_z}xy, \tag{53}$$

$$u_y = \frac{1}{2}\frac{(1+\alpha_o)}{1+\alpha_o+\psi_o\delta}\frac{M}{\tilde{E}I_z}(\psi_o x^2+\alpha_o y^2). \tag{54}$$

It should be noted that if p_0 and m_{sz} do not satisfy (42); these solutions do not hold and pure bending cannot take place.

5. Reduction of solution to isotropic continuum

When reducing orthotropic continuum to the isotropic one, all the material parameters are reduced to just two as

$$E_i = E,$$

$$n_{ij} = n,$$

for *i, j = x, y, z.* This means that orthotropic constants $\tilde{E}$, ψ_o and α_o from (17) take the following form:

$$\tilde{E} = \frac{E}{1-n^2},$$

$$\psi_o = 1, \tag{55}$$

$$\alpha_o = \frac{n}{1-n}.$$

When equations (55) are substituted in (47) and (48), the analytical solution for pure bending of isotropic micropolar continuum follows as

$$\sigma_{xx} = -\frac{1}{1+(1-n)\delta}\frac{M}{I_z}y, \tag{56}$$

$$\mu_{zx} = \frac{(1-n)\delta}{1+(1-n)\delta}\frac{M}{A}. \tag{57}$$

In the same manner, by substituting (55) in (49), (50) and (51), the analytical solution for the strain components is obtained as

$$\epsilon_{xx} = -\frac{1-n^2}{1+(1-n)\delta}\frac{M}{EI_z}y, \tag{58}$$

$$\epsilon_{yy} = \frac{n}{1-n}\frac{1-n^2}{1+(1-n)\delta}\frac{M}{EI_z}y, \tag{59}$$

$$\kappa_{zx} = \frac{1-n^2}{1+(1-n)\delta}\frac{M}{EI_z}. \tag{60}$$

Finally, by substituting (55) in (52), (53) and (54), the isotropic micropolar kinematic fields for the case of pure bending are obtained as

$$\varphi_z = \frac{1-n^2}{1+(1-n)\delta}\frac{M}{EI_z}x, \tag{61}$$

$$u_x = -\frac{1-n^2}{1+(1-n)\delta}\frac{M}{EI_z}xy, \tag{62}$$

$$u_y = \frac{1}{2}\frac{1-n^2}{1+(1-n)\delta}\frac{M}{EI_z}(\psi_o x^2 + \alpha_o y^2). \tag{63}$$

It can be verified that equations derived in this section are equal to those derived in [1].

6. Conclusion

It can be seen that the reduced orthotropic micropolar material model presented demands seven material parameters to be known: two Young's moduli, four Poisson's coefficients and one characteristic length for bending. While this model is not fully orthotropic, it can be used as a tool for identification of the characteristic length, which will be shown in a separate work. Its advantage is that it enables such identification without drastically increasing the number of other micropolar engineering parameters. While the engineering parameters known from classical elasticity theory can be easily identified for a given microstructure, the methods for identifying the micropolar parameters are not yet generally recognised.

The disadvantage of this model is that, as can be seen from the cited references, a larger number of characteristic lengths for bending should be considered in order to properly model micropolar materials. Furthermore, the model is not universally applicable, and can really only be used to identify the characteristic length for bending.

Finally, it is shown that by including isotropic material parameters in their orthotropic equivalents, analytical solutions known from the literature for this type of problem are obtained.

Acknowledgement. *This study was part of activities within the scientific project Fixed-Pole Concept in Numerical Modeling of Cosserat Continuum (HRZZ-IP-2018-01-1732), which is financed by the Croatian Science Foundation and university support of the University of Rijeka Computational and Experimental Procedures for Assessment of Material Parameters in the Cosserats continuum (uniri-tehnic-18-248 1415).*

References

[1] Gauthier, R.D., Jahsman, W.E. (1975) A Quest for Micropolar Elastic Constants. Journal of Applied Mechanics, 42(2), 369-374.

[2] Alemi, B., Shodja, H.M. (2021) Effective Moduli and Characteristic Lengths of Micropolar Media with Dense Periodic Distribution of Ellipsoidal Nano-/Micro-Inhomogeneities. European Journal of Mechanics /A Solids, 85.

[3] Grbčić, S. (2018) Linked Interpolation and Strain Invariance in Finite-Element Modelling of Micropolar Continuum. PhD thesis. University of Rijeka, Faculty of Civil Engineering

[4] Lakes, R.S. (1983) Size Effect and Micromechanics of a Porous Solid. Journal of Material Science, 18, 2572-2580.

[5] Bouyge, F., Jasiuk, I. Boccara, S., Ostoja-Starzewski, M. (2002) A Micromechanically Based Couple-Stress Model of an Elastic Orthotropic Two-Phase Composite. European Journal of Mechanics /A Solids, 21, 465-481.

[6] Ieșan, D. (1973) The Plane Micropolar Strain of Orthotropic Elastic Solids. Archives of Mechanics, 25(3), 547-561.

[7] Jelenić G. (2021) On Pure Bending in Non-Linear Elasticity: A Circular Closed-Form 2d Solution for Semi-Linear Orthotropic Material. European Journal of Mechanics / A Solids, 90.

[8] Timoshenko, S., Goodier J.N. (1951) Theory of Elasticity. New York. McGraw-Hill book company.