Accurate Numerical Solutions for Standard Skew Plate Benchmark Problems

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ACCURATE NUMERICAL SOLUTIONS FOR STANDARD SKEW PLATE BENCHMARK PROBLEMS

TOČNA NUMERIČKA RJEŠENJA ZA STANDARDNE TESTNE PROBLEME KOSIH PLOČA

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Abstract

Morley's skew plate and Razzaque's skew plate are two standard benchmark problems used for the performance evaluation of plate finite elements. The reference solutions for these problems are given for the thin plate case only, and various plate finite elements do not converge to these solutions exactly. Accurate numerical solutions for these skew plate problems are proposed in this paper, which are based on the finite element analysis on very dense meshes for thin to thick plate cases.

Key words: *Morley's skew plate, Razzaque's skew plate, plate finite elements, finite element analysis, accurate solutions, convergence, thin and thick plates*

Sažetak

Morleyjeva kosa ploča i Razzaqueova kosa ploča su dva standardna testna problema koja se koriste za evaluaciju performansi konačnih elemenata ploča. Referentna rješenja za ove probleme su dana samo za slučajeve tankih ploča, a i razni konačni elementi ploča ne konvergiraju točno ka tim rješenjima. U ovome radu predlažu se točna numerička rješenja za ove probleme kosih ploča, koja su bazirana na analizi konačnih elemenata na vrlo gustim mrežama za slučajeve tankih do debelih ploča.

Ključne riječi: *Morleyjeva kosa ploča, Razzaqueova kosa ploča, konačni elementi ploča, analiza konačnih elemenata, točna rješenja, konvergencija, tanke i debele ploče*

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1. Introduction

The last step in the finite element development process is convergence assessment and performance evaluation. For the convergence assessment, the elements are subjected to the so-called *patch test*, while for the performance evaluation, the elements are subjected to a set of standard benchmark problems. Even though the *patch test* checks whether the element is theoretically capable of converging to the correct solution of the real problem with the finite element mesh refinement, it does not give the information on how fast the convergence rate is. As the elements with higher performance are desired for practical purposes, this information is very important. Therefore, the elements are subjected to a set of standard benchmark problems, which can clearly indicate their performance level.

Morley's skew plate [1] and Razzaque's skew plate [2] are two benchmark problems commonly used for the performance evaluation of plate finite elements. Central displacement and moment values are checked and compared with the reference ones given by the authors, which are based on the classical (thin) plate theory, and are given either analytically [1] or numerically [2]. However, the reference values are to a certain extent offset from the ones obtained by various plate finite elements on dense meshes, as it can be observed in Ref. [3]. This is not to be unexpected as the error exists in in both analytical and numerical solutions, and because the reference solutions are given for the limiting thin plate cases only, i.e., shear strains contribution is disregarded. Due to this discrepancy, this paper aims to propose more accurate numerical solutions, which in turn would allow for more accurate performance evaluations. In addition, solutions for different thin to thick plate cases are also proposed, which would allow for better performance evaluation between plate finite elements that recognise shear strains contribution, i.e., shear deformable plate elements.

2. Numerical analysis of standard skew plate benchmark problems

2.1. Finite element analysis on very dense meshes

Finite element analysis of the standard skew plate benchmark problems on very dense meshes is performed in order to obtain accurate numerical solutions. The finite element analysis is carried out in the Finite Element Analysis Program (FEAP) [4]. Different three-node shear deformable plate finite elements are used in the analysis, viz., T3-2LIM by Auricchio and Taylor [5] (default plate finite element in FEAP), ARS-T9 by Soh *et al.*[6], T3-U2 by Ribarić and Jelenić [7] and T3-LSI by Grbac and Ribarić [3], so the assessment of the accurate solution is as objective as possible, as some elements, depending on the analysed case, may exhibit better convergence rate than others.

The T3-2LIM element is a mixed formulation element with quadratic linked interpolation, in which the rotational fields are enriched with internal degrees of freedom. The shear strain fields are assumed constant and are interpolated independently. The ARS-T9 element utilises the Timoshenko beam element with problem dependent cubic linked interpolation in a way that the expressions for the shear strain and the rotations are applied along the each of the plate element sides. The T3-U2 element is a classical displacement-based element with quadratic linked interpolation. Lastly, the T3-LSI element is also a linked interpolation element that shares the same shear strains expressions with the ARS-T9 element, but its rotational fields have a specific independent interpolation. All the aforementioned elements pass the constant bending *patch test* regardless of the element size, so the convergence to the correct solution can be reliably expected.

The skew plate models are analysed as whole plates and are discretised with the following meshes: 128 x 128, 256 x 256, 512 x 512 and 1024 x 1024. The numbers indicate the number of divisions throughout the entire plate domain in both plate edge directions. All these meshes are considered for better assessment of the accurate solution, as the information on which finite element is closest to the correct solution would be indicated by their convergence rate.

The plates are uniformly loaded with the loading intensity q = 1, and the plate properties being E = 10.92, ν = 0.3 and k = 5/6, where E is Young's modulus, ν is Poisson's ratio and k is the shear correction factor. The finite element models have three unknown degrees of freedom per node, viz., transverse displacement w and two mutually orthogonal section rotations θ_x and θ_y, as shear deformable plate finite elements are used in the analysis.

2.2. Morley's skew plate

Morley analysed this skew plate problem in Ref. [1]. The plate has a small acute angle of 30°, and the edges with the length L = 100, as shown in Figure 1. It is simply supported on its edges, with rotations in both coordinate directions being unrestrained (w = 0, $\theta_x \neq 0$ and $\theta_y \neq 0$). This problem is specific as theoretically infinite principal moments occur in the obtuse corners of the plate, one having a positive and the other having a negative sign. This unique property results in a slower convergence to the correct solution in the finite element analysis.

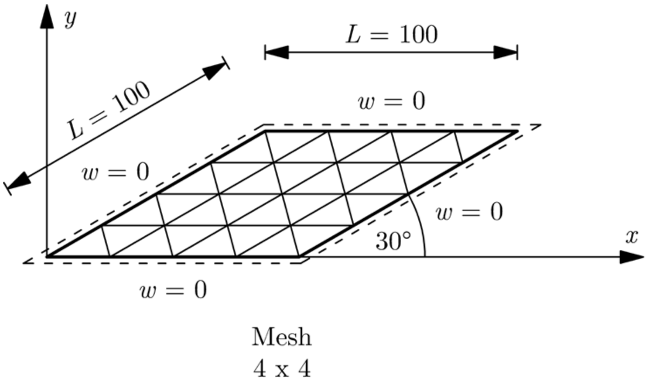

Figure 1. *Morley's skew plate model*

Morley has also provided an analytical solution for this problem for the limiting Kirchhoff-Germain assumptions (thin plate solution). Normalised central transverse displacement $w_c^N = w_c/(10^{-3}qL^4/D)$ is 0.408, while normalised principal moments $M_{1,2}^N = M_{1,2}/(10^{-2}qL^2)$ are 1.91 and 1.08. D is the plate bending rigidity given as $D = Eh^3/[12(1-\nu^2)]$, in which h is the plate thickness. It is important to note that these values are not exact, as they are obtained from a series solution that has a finite number of terms.

The results of the finite element analysis are shown in Tables 1-3, based on which accurate solutions are proposed in Table 4. For the L/h = 1000 case, the convergence behaviour of the T3-2LIM, ARS-T9 and T3-LSI elements leads to inconclusive assessment of the accurate solutions. Hence, the proposed accurate solutions are based on the T3-U2 element results due to its clear convergence behaviour. Moreover, due to the T3-U2 element's simple displacement-based formulation, adopting its results as accurate on such dense meshes is consistent with the concept of the finite element method itself. Table 5 shows reference solution (r_s) error with respect to the proposed accurate solutions (a_s) as $|a_s-r_s|/a_s$. By increasing the L/h ratio, the finite element results should approach the analytical values more closely, even though they are not exact.

Table 1. *Normalised central transverse displacement* w_c^N *of the Morley's skew plate*

L/h = 1000				
Mesh	**T3-2LIM**	**ARS-T9**	**T3-U2**	**T3-LSI**
128 x 128	0.41407	0.41313	0.40208	0.41441
256 x 256	0.41320	0.41227	0.41011	0.41335
512 x 512	0.41314	0.41230	0.41234	0.41313
1024 x 1024	0.41360	0.41315	0.41338	0.41354
L/h = 100				
Mesh	**T3-2LIM**	**ARS-T9**	**T3-U2**	**T3-LSI**
128 x 128	0.42346	0.42260	0.42304	0.42333
256 x 256	0.42433	0.42407	0.42422	0.42428
512 x 512	0.42468	0.42461	0.42465	0.42467
1024 x 1024	0.42479	0.42477	0.42478	0.42479
L/h = 10				
Mesh	**T3-2LIM**	**ARS-T9**	**T3-U2**	**T3-LSI**
128 x 128	0.51759	0.51755	0.51754	0.51760
256 x 256	0.51763	0.51763	0.51762	0.51764
512 x 512	0.51765	0.51765	0.51764	0.51765
1024 x 1024	0.51765	0.51765	0.51765	0.51765

Table 2. *Normalised principal moment* M_1^N *of the Morley's skew plate obtained in Gauss point that is closest to the centre of the plate*

L/h = 1000				
Mesh	**T3-2LIM**	**ARS-T9**	**T3-U2**	**T3-LSI**
128 x 128	1.924	1.921	1.889	1.925
256 x 256	1.922	1.919	1.913	1.922
512 x 512	1.922	1.919	1.919	1.922
1024 x 1024	1.923	1.922	1.922	1.923
L/h = 100				
Mesh	**T3-2LIM**	**ARS-T9**	**T3-U2**	**T3-LSI**
128 x 128	1.951	1.948	1.950	1.951
256 x 256	1.954	1.953	1.954	1.954
512 x 512	1.955	1.955	1.955	1.955
1024 x 1024	1.956	1.956	1.956	1.956
L/h = 10				
Mesh	**T3-2LIM**	**ARS-T9**	**T3-U2**	**T3-LSI**
128 x 128	2.064	2.064	2.064	2.064
256 x 256	2.064	2.064	2.064	2.064
512 x 512	2.064	2.064	2.064	2.064
1024 x 1024	2.064	2.064	2.064	2.064

***Table 3.** Normalised principal moment M_2^N of the Morley's skew plate obtained in Gauss point that is closest to the centre of the plate*

L/h = 1000				
Mesh	**T3-2LIM**	**ARS-T9**	**T3-U2**	**T3-LSI**
128 x 128	1.108	1.105	1.064	1.110
256 x 256	1.105	1.102	1.094	1.106
512 x 512	1.105	1.102	1.102	1.105
1024 x 1024	1.107	1.105	1.106	1.107
L/h = 100				
Mesh	**T3-2LIM**	**ARS-T9**	**T3-U2**	**T3-LSI**
128 x 128	1.138	1.136	1.137	1.138
256 x 256	1.141	1.141	1.141	1.141
512 x 512	1.143	1.143	1.143	1.143
1024 x 1024	1.143	1.143	1.143	1.143
L/h = 10				
Mesh	**T3-2LIM**	**ARS-T9**	**T3-U2**	**T3-LSI**
128 x 128	1.205	1.205	1.205	1.205
256 x 256	1.205	1.205	1.205	1.205
512 x 512	1.205	1.205	1.205	1.205
1024 x 1024	1.205	1.205	1.205	1.205

***Table 4.** Proposed accurate solutions for the Morley's skew plate*

L/h	w_c^N	M_1^N	M_2^N
1000	0.4134	1.922	1.106
100	0.4248	1.956	1.143
10	0.5177	2.064	1.205
Ref. [1]	0.408	1.91	1.08

***Table 5.** Reference solution error with respect to the proposed accurate solutions for the Morley's skew plate*

L/h	w_c^N	M_1^N	M_2^N
1000	1,31%	0,62%	2,35%
100	3,95%	2,35%	5,51%
10	21,19%	7,46%	10,37%

2.3. Razzaque's skew plate

This skew plate problem was introduced by Razzaque in Ref. [2]. The plate has an acute angle of 60°, and the edges with the length L = 100, as shown in Figure 2. Additionally, it is simply supported on two opposite edges (w = 0 and θ_y = 0).

Razzaque adopted a thin plate finite difference solution on a 16 x 16 mesh as a reference solution, which is given here in terms of normalised central transverse displacement $w_c^N = w_c/(10^{-2}qL^4/D) = 0.7945$ and normalised central moment $M_{y,c}^N = M_{y,c}/(10^{-1}qL^2) = 0.9589$. The results of the finite element analysis are shown in Tables 6-7, based on which accurate solutions are proposed in Table 8. Reference solution (r_s) error with respect to the proposed accurate solutions (a_s) given as $|a_s\text{-}r_s|/a_s$ is shown in Table 9. Additionally, the finite elements' results and errors with respect to the proposed accurate solutions on a 16 x 16 mesh are shown in Table 10 as well, since the reference solution is given on the same mesh.

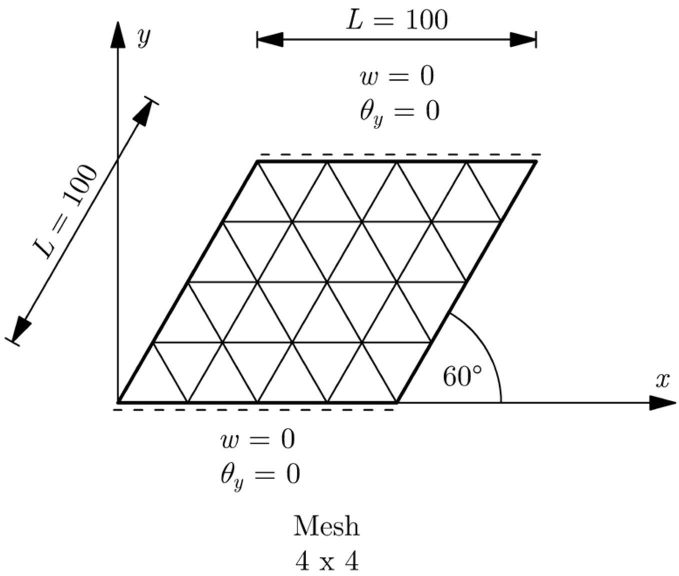

Figure 2. *Razzaque's skew plate model*

Table 6. *Normalised central transverse displacement w_c^N of the Razzaque's skew plate*

***L/h* = 1000**				
Mesh	**T3-2LIM**	**ARS-T9**	**T3-U2**	**T3-LSI**
128 x 128	0.79137	0.79095	0.79080	0.79136
256 x 256	0.79123	0.79100	0.79104	0.79123
512 x 512	0.79119	0.79105	0.79110	0.79118
1024 x 1024	0.79120	0.79113	0.79116	0.79118
***L/h* = 100**				
Mesh	**T3-2LIM**	**ARS-T9**	**T3-U2**	**T3-LSI**
128 x 128	0.79335	0.79288	0.79309	0.79330
256 x 256	0.79356	0.79342	0.79348	0.79354
512 x 512	0.79364	0.79361	0.79362	0.79363
1024 x 1024	0.79366	0.79366	0.79366	0.79366
L/h = 10				
Mesh	**T3-2LIM**	**ARS-T9**	**T3-U2**	**T3-LSI**
128 x 128	0.84374	0.84365	0.84368	0.84374
256 x 256	0.84382	0.84379	0.84380	0.84381
512 x 512	0.84384	0.84383	0.84383	0.84384
1024 x 1024	0.84384	0.84384	0.84384	0.84384

Table 7. *Normalised moment M_y^N of the Razzaque's skew plate obtained in Gauss point that is closest to the centre of the plate*

***L/h* = 1000**				
Mesh	**T3-2LIM**	**ARS-T9**	**T3-U2**	**T3-LSI**
128 x 128	0.9602	0.9599	0.95 98	0.9602
256 x 256	0.9601	0.9600	0.9600	0.9601
512 x 512	0.9601	0.9600	0.9600	0.9601
1024 x 1024	0.9601	0.9601	0.9601	0.9601
***L/h* = 100**				
Mesh	**T3-2LIM**	**ARS-T9**	**T3-U2**	**T3-LSI**
128 x 128	0.9615	0.9611	0.9613	0.9614
256 x 256	0.9617	0.9616	0.9616	0.9616
512 x 512	0.9617	0.9617	0.9617	0.9617
1024 x 1024	0.9617	0.9617	0.9617	0.9617
***L/h* = 10**				
Mesh	**T3-2LIM**	**ARS-T9**	**T3-U2**	**T3-LSI**
128 x 128	0.9805	0.9804	0.9805	0.9805
256 x 256	0.9806	0.9805	0.9805	0.9806
512 x 512	0.9806	0.9806	0.9806	0.9806
1024 x 1024	0.9806	0.9806	0.9806	0.9806

Table 8. *Proposed accurate solutions for the Razzaque's skew plate*

L/h	w_c^N	$M_{y,c}^N$
1000	0.7912	0.9601
100	0.7937	0.9617
10	0.8438	0.9806
Ref. [2]	0.7945	0.9589

Table 9. *Reference solution error with respect to the proposed accurate solutions for the Razzaque's skew plate*

L/h	w_c^N	$M_{y,c}^N$
1000	0,42%	0,12%
100	0,10%	0,29%
10	5,84%	2,21%

Table 10. *Results and errors with respect to the proposed accurate solutions on a 16 x 16 mesh for the Razzaque's skew plate*

L/h = 1000					
	T3-2LIM	ARS-T9	T3-U2	T3-LSI	Ref. [2]
w_c^N	0.79316	0.78805	0.73892	0.79281	0.7945
$M_{y,c}^N$	0.9583	0.9544	0.8996	0.9560	0.9589
w_c^N error	0,25%	0,40%	6,61%	0,20%	0,42%
$M_{y,c}^N$ error	0,19%	0,59%	6,30%	0,43%	0,12%

3. Conclusion

The obtained finite element solutions for different plate finite elements are in a very good agreement, as they all appear to converge to the same values, apart from the inconclusive Morley L/h = 1000 case. This makes the assessments of the accurate solutions rather straightforward. Moreover, all finite element solutions are in the exact agreement for the L/h = 10 cases on the densest mesh (1024 x 1024). However, the reference solutions are not in agreement with the obtained finite element solutions.

Since the proposed accurate solutions are based on the finite element analysis on very dense meshes, it is suggested that they are adopted instead as the reference solutions for the performance evaluation of plate finite elements in the future. Additionally, the reference solutions are given for the limiting thin plate cases only, whereas the proposed accurate solutions include thin to thick plate cases.

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